

# FIXED POINT THEOREMS AND APPLICATIONS IN THEORY OF GAMES

Monica Patriche

*University of Bucharest Faculty of Mathematics and Computer Science  
14 Academiei Street  
010014 Bucharest, Romania  
monica.patriche@yahoo.com*

---

## Abstract

We introduce the notions of weakly  $*$ -concave and weakly naturally quasi-concave correspondence and prove fixed point theorems and continuous selection theorems for these kind of correspondences. As applications in the game theory, by using a technique based on a continuous selection, we establish new existence results for the equilibrium of the abstract economies. The constraint correspondences are weakly naturally quasi-concave. We show that the equilibrium exists without continuity assumptions.

*Keywords:* weakly naturally quasi-concave correspondence, fixed point theorem, continuous selection, abstract economy, equilibrium.

---

## 1. Introduction

It is known that the theory of correspondences has very widely developed and produced many applications, especially during the last few decades. Most of these applications concern fixed point theory and game theory. The fixed point theorems are closely connected with convexity. A considerable number of papers devotes to correspondences on nonconvex and noncompact domains (see e.g. [16], [17], [18]) or to correspondences without convex values and continuity ([5]).

The aim of this paper is to introduce the notions of weakly  $*$ -concave and weakly naturally quasi-concave correspondence and prove fixed point theorems and continuous selection theorems for these kind of correspondences. We also define the correspondences with WNQS and e-WNQS property.

The applications concern the equilibrium theory: we establish new existence results for the equilibrium of the abstract economies. The constraint correspondences are weakly concave-like or have WNQS, respectively e-WNQS property.

For the reader's convenience, we review the main results in the equilibrium theory, emphasizing that most authors have studied the existence of equilibrium for abstract economies with preferences represented as correspondences which have continuity properties. We mention here the results obtained by W. Shafer and H. Sonnenschein [14], which concern economies with finite dimensional commodity space and preference correspondences having an open graph. N. C. Yannelis and N. D. Prahbakar [21] used selection theorems and fixed-point theorems for correspondences with open lower sections defined on infinite dimensional strategy spaces. Some authors developed the theory of continuous selections of correspondences and gave numerous applications in game theory. Michael's selection theorem [11] is well-known and basic in many applications. In [3,4], F. Browder firstly used a continuous selection theorem to prove Fan-Browder fixed point theorem. Later, N. C. Yannelis and N. D. Prabhakar [21], H. Ben-El-Mechaiekh [1], X. Ding, W. Kim and K. Tan [6], C. Horvath [9], T. Husain and E. Taradfar [10], S. Park [12],[13], X. Wu [19], X. Wu and S. Shen [20], Z. Yu and L. Lin [22] and many others established several continuous selection theorems with applications.

In this paper, we show that an equilibrium for an abstract economy exists without continuity assumptions. By using a technique based on a continuous selection, we prove the new equilibrium existence theorem for an abstract economy.

The paper is organized in the following way: Section 2 contains preliminaries and notations. The fixed point and the selection theorem are presented in Section 3. The equilibrium theorems are stated in Section 4.

## 2. PRELIMINARIES AND NOTATIONS

Throughout this paper, we shall use the following notations and definitions:

Let  $A$  be a subset of a topological space  $X$ .

1.  $2^A$  denotes the family of all subsets of  $A$ .
2.  $\text{cl } A$  denotes the closure of  $A$  in  $X$ .
3. If  $A$  is a subset of a vector space,  $\text{co}A$  denotes the convex hull of  $A$ .

4. If  $F, T : A \rightarrow 2^X$  are correspondences, then  $\text{co}T, \text{cl } T, T \cap F : A \rightarrow 2^X$  are correspondences defined by  $(\text{co}T)(x) = \text{co}T(x)$ ,  $(\text{cl}T)(x) = \text{cl}T(x)$  and  $(T \cap F)(x) = T(x) \cap F(x)$  for each  $x \in A$ , respectively.
  5. The graph of  $T : X \rightarrow 2^Y$  is the set  $\text{Gr}(T) = \{(x, y) \in X \times Y \mid y \in T(x)\}$
  6. The correspondence  $\overline{T}$  is defined by  $\overline{T}(x) = \{y \in Y : (x, y) \in \text{cl}_{X \times Y} \text{Gr}T\}$  (the set  $\text{cl}_{X \times Y} \text{Gr}(T)$  is called the adherence of the graph of  $T$ ).
- It is easy to see that  $\text{cl}T(x) \subset \overline{T}(x)$  for each  $x \in X$ .

**Lemma 1.** (see [23]) *Let  $X$  be a topological space,  $Y$  be a non-empty subset of a topological vector space  $E$ ,  $\beta$  be a base of the neighborhoods of  $0$  in  $E$  and  $A : X \rightarrow 2^Y$ . For each  $V \in \beta$ , let  $A_V : X \rightarrow 2^Y$  be defined by  $A_V(x) = (A(x) + V) \cap Y$  for each  $x \in X$ . If  $\hat{x} \in X$  and  $\hat{y} \in Y$  are such that  $\hat{y} \in \bigcap_{V \in \beta} \overline{A_V}(\hat{x})$ , then  $\hat{y} \in \overline{A}(\hat{x})$ .*

**Definition 1.** *Let  $X, Y$  be topological spaces and  $T : X \rightarrow 2^Y$  be a correspondence*

1.  $T$  is said to be *upper semicontinuous* if for each  $x \in X$  and each open set  $V$  in  $Y$  with  $T(x) \subset V$ , there exists an open neighborhood  $U$  of  $x$  in  $X$  such that  $T(y) \subset V$  for each  $y \in U$ .
2.  $T$  is said to be *lower semicontinuous* if for each  $x \in X$  and each open set  $V$  in  $Y$  with  $T(x) \cap V \neq \emptyset$ , there exists an open neighborhood  $U$  of  $x$  in  $X$  such that  $T(y) \cap V \neq \emptyset$  for each  $y \in U$ .
3.  $T$  is said to have *open lower sections* if  $T^{-1}(y) := \{x \in X : y \in T(x)\}$  is open in  $X$  for each  $y \in Y$ .

**Lemma 2.** (see [24]). *Let  $X$  be a topological space,  $Y$  be a topological linear space, and let  $A : X \rightarrow 2^Y$  be an upper semicontinuous correspondence with compact values. Assume that the sets  $C \subset Y$  and  $K \subset Y$  are closed and respectively compact. Then  $T : X \rightarrow 2^Y$  defined by  $T(x) = (A(x) + C) \cap K$  for all  $x \in X$  is upper semicontinuous.*

We present the following types of generalized convex functions and correspondences.

**Definition 2.** (see [15]) Let  $X$  be a convex set in a real vector space, and let  $Z$  be an ordered t.v.s, with a pointed convex cone  $C$ . A vector-valued  $f : X \rightarrow Z$  is said to be *natural quasi  $C$ -convex* on  $X$  if  $f(\lambda x_1 + (1 - \lambda)x_2) \in \text{co}\{f(x_1), f(x_2)\} - C$  for every  $x_1, x_2 \in X$  and  $\lambda \in [0, 1]$ . This condition is equivalent with the following condition: there exists  $\mu \in [0, 1]$  such that  $f(\lambda x_1 + (1 - \lambda)x_2) \leq_C \mu f(x_1) + (1 - \mu)f(x_2)$ , where  $x \leq_C y \Leftrightarrow y - x \in C$ .

A vector-valued function  $f$  is said to be *natural quasi  $C$ -concave* on  $X$  if  $-f$  is natural quasi  $C$ -convex on  $X$ .

**Definition 3.** (see [26]) Let  $E_1, E_2$  and  $Z$  be real Hausdorff topological vector spaces,  $C \subset Z$  be a closed convex pointed cone with  $\text{int}C \neq \emptyset$ ; let  $X$  be a nonempty convex subset of  $E_1$ ,  $T : X \rightarrow 2^Z$  be a correspondence.  $T$  is said to be *naturally  $C$ -quasi-concave* on  $X$ , if for any  $x_1, x_2 \in X$  and  $\lambda \in [0, 1]$ ,  $\text{co}(T(x_1), T(x_2)) \subset T(\lambda x_1 + (1 - \lambda)x_2) - C$ .

Let  $\Delta_{n-1} = \left\{ (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n : \sum_{i=1}^n \lambda_i = 1 \text{ and } \lambda_i \geq 0, i = 1, 2, \dots, n \right\}$  be the standard  $(n-1)$ -dimensional simplex in  $\mathbb{R}^n$ .

**Definition 4.** (see [5]) Let  $X$  be a non-empty convex subset of a topological vector space  $E$  and  $Y$  be a non-empty subset of  $E$ . The correspondence  $T : X \rightarrow 2^Y$  is said to have *weakly convex graph* (in short, it is a *WCG correspondence*) if for each finite set  $\{x_1, x_2, \dots, x_n\} \subset X$ , there exists  $y_i \in T(x_i)$ , ( $i = 1, 2, \dots, n$ ) such that

$$(1.1) \quad \text{co}(\{(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)\}) \subset \text{Gr}(T)$$

The relation (1.1) is equivalent to

$$(1.2) \quad \sum_{i=1}^n \lambda_i y_i \in T\left(\sum_{i=1}^n \lambda_i x_i\right) \quad (\forall (\lambda_1, \lambda_2, \dots, \lambda_n) \in \Delta_{n-1}).$$

We introduce the concept of weakly naturally quasi-concave correspondence.

**Definition 5.** Let  $X$  be a nonempty convex subset of a topological vector space  $E$  and  $Y$  be a nonempty subset of a topological vector space  $F$ . The correspondence  $T : X \longrightarrow 2^Y$  is said to be weakly naturally quasi-concave (WNQ) if for each  $n$  and for each finite set  $\{x_1, x_2, \dots, x_n\} \subset X$ , there exists  $y_i \in T(x_i)$ ,  $(i = 1, 2, \dots, n)$  and  $g = (g_1, g_2, \dots, g_n) : \Delta_{n-1} \rightarrow \Delta_{n-1}$  a function with  $g_i$  continuous,  $g_i(1) = 1$  and  $g_i(0) = 0$  for each  $i = 1, 2, \dots, n$ , such that for every  $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \Delta_{n-1}$ , there exists  $y = \sum_{i=1}^n g_i(\lambda_i) y_i \in T(\sum_{i=1}^n \lambda_i x_i)$ .

**Remark 1.** If  $g_i(\lambda_i) = \lambda_i$  for each  $i \in (1, 2, \dots, n)$  and  $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \Delta_{n-1}$ , we get a correspondence with weakly convex graph, as it is defined by Ding and He Yiran in [5]. In the same time, the weakly naturally quasi-concavity is a weakening of the notion of naturally  $C$ -quasi-concavity with  $C = \{0\}$ .

**Remark 2.** If  $T$  is a single valued mapping, then it must be natural quasi  $C$ -concave for  $C = \{0\}$ .

**Example 1.** Let  $T : [0, 4] \rightarrow 2^{[-2, 2]}$  be defined by  $T(x) = \begin{cases} [0, 2] & \text{if } x \in [0, 2); \\ [-2, 0] & \text{if } x = 2; \\ (0, 2] & \text{if } x \in (2, 4]. \end{cases}$

$T$  is neither upper semicontinuous, nor lower semicontinuous in 2.  $T$  also has not weakly convex graph, since if we consider  $n = 2$ ,  $x_1 = 1$  and  $x_2 = 3$ , we have that  $\text{co}\{(1, y_1), (3, y_2)\} \not\subseteq \text{Gr}T$  for every  $y_1 \in T(x_1), y_2 \in T(x_2)$ .

We shall prove that  $T$  is a weakly naturally quasi-concave correspondence.

1) Let's consider first  $n = 2$ .

a) If  $x_1, x_2 \in [0, 2)$  and  $x_1, x_2 \in (2, 4]$ , there exists  $y_1 = 2 \in T(x_1), y_2 = 2 \in$

$T(x_2)$  and  $g_i(\lambda_i) = \lambda_i$ ,  $i = 1, 2$  such that for each  $(\lambda_1, \lambda_2)$  with the property

that  $\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_1 + \lambda_2 = 1$ , there exists  $y = \sum_{i=1}^2 g_i(\lambda_i) y_i \in$

$T(\sum_{i=1}^2 \lambda_i x_i)$ .

b) If  $x_1 \in [0, 2)$  and  $x_2 \in (2, 4]$ , there exists  $\lambda_1^* \neq 0$  such that  $\lambda_1^* x_1 + (1 - \lambda_2^*) x_2 = 2$ .

Let's consider  $g_i : [0, 1] \rightarrow [0, 1]$  continuous functions such that  $g_i(1) = 1$ ,  $g_i(0) = 0$  for each  $i = 1, 2$  and  $g_1(\lambda_1) + g_2(\lambda_2) = 1$  if  $\lambda_1 + \lambda_2 = 1$ , defined by

$$g_1(\lambda_1) = \begin{cases} \frac{1}{\lambda_1^*} \lambda_1 & \text{if } \lambda_1 \in [0, \lambda_1^*]; \\ 1 & \text{if } \lambda_1 \in [\lambda_1^*, 1] \end{cases} \quad \text{and} \\ g_2(\lambda_2) = \begin{cases} 0 & \text{if } \lambda_2 \in [0, 1 - \lambda_1^*]; \\ 1 - \frac{1}{\lambda_1^*} + \frac{1}{\lambda_1^*} \lambda_2 & \text{if } \lambda_2 \in (1 - \lambda_1^*, 1]. \end{cases}$$

There exists  $y_1 = 0$  and  $y_2 = 2$  such that

b1) for  $\lambda_1 \in [0, \lambda_1^*)$  and  $\lambda_2 = 1 - \lambda_1$ ,  $x = \lambda_1 x_1 + \lambda_2 x_2 \in (2, x_2]$ , then  $T(x) = (0, 2]$  and  $y = g_1(\lambda_1)y_1 + g_2(\lambda_2)y_2 = \frac{1}{\lambda_1^*} \lambda_1 y_1 + (1 - \frac{1}{\lambda_1^*} \lambda_1)y_2 = (1 - \frac{1}{\lambda_1^*} \lambda_1)2 \in (0, 2] = T(\lambda_1 x_1 + \lambda_2 x_2)$ ;

b2) for  $\lambda_1 \in (\lambda_1^*, 1]$ , and  $\lambda_2 = 1 - \lambda_1$ ,  $x = \lambda_1 x_1 + \lambda_2 x_2 \in [x_1, 2)$ , then  $T(x) = [0, 2]$  and  $y = g_1(\lambda_1)y_1 + g_2(\lambda_2)y_2 = 1 \times 0 + 0 \times 2 = 0 \in T(\lambda_1 x_1 + \lambda_2 x_2)$ ;

b3) If  $\lambda_1 = \lambda_1^*$ ,  $\lambda_2 = 1 - \lambda_1^*$ ,  $x = \lambda_1 x_1 + \lambda_2 x_2 = 2$ , then  $T(x) = [-2, 0]$  and  $y = g_1(\lambda_1)y_1 + g_2(\lambda_2)y_2 = 1 \times 0 + 0 \times 2 = 0 \in T(2)$ ;

c) If  $x_1 \in [0, 2)$  and  $x_2 = 2$ , there exists  $y_1 = 2$ ,  $y_2 = 0$  and the continuous functions  $g_i : [0, 1] \rightarrow [0, 1]$  with  $g_i(1) = 1$ ,  $g_i(0) = 0$  for each  $i = 1, 2$  and

$g_1(\lambda_1) + g_2(\lambda_2) = 1$  if  $\lambda_1 + \lambda_2 = 1$  such that

c1) for  $\lambda_1 \in (0, 1]$  and  $\lambda_2 = 1 - \lambda_1$ ,  $x = \lambda_1 x_1 + \lambda_2 x_2 \in [x_1, x_2)$ , then  $T(x) = [-2, 0]$  and  $y = g_1(\lambda_1)y_1 + g_2(\lambda_2)y_2 = g_1(\lambda_1) \times 2 + g_2(\lambda_2) \times 0 = g_1(\lambda_1) \times 2 \in T(x)$ ;

c2) for  $\lambda_1 = 0$  and  $\lambda_2 = 1$ ,  $x = 2$ , then  $T(2) = [-2, 0]$  and  $y = g_1(0)y_1 + g_2(1)y_2 = 0 \times 2 + 1 \times 0 = 0 \in T(2)$ ;

d) If  $x_1 = 2$  and  $x_2 \in (2, 4]$ , there exists  $y_1 = 0$ ,  $y_2 = 2$  and the continuous functions  $g_i : [0, 1] \rightarrow [0, 1]$  with  $g_i(1) = 1$ ,  $g_i(0) = 0$  for each  $i = 1, 2$  and

$g_1(\lambda_1) + g_2(\lambda_2) = 1$  if  $\lambda_1 + \lambda_2 = 1$  such that

d1) for  $\lambda_1 = 1$  and  $\lambda_2 = 0$ ,  $x = 2$ , then  $T(2) = [-2, 0]$  and  $y = g_1(1)y_1 + g_2(0)y_2 = 1 \times 0 + 0 \times 2 = 0 \in T(2)$ ;

d2) for  $\lambda_1 \in [0, 1)$  and  $\lambda_2 = 1 - \lambda_1$ ,  $x = \lambda_1 x_1 + \lambda_2 x_2 \in (x_1, x_2]$ , then  $T(x) = (0, 2]$  and  $y = g_1(\lambda_1)y_1 + g_2(\lambda_2)y_2 = g_1(\lambda_1) \times 0 + g_2(\lambda_2) \times 2 = g_2(\lambda_2) \times 2 \in (0, 2] = T(x)$ .

2) The case  $n > 2$  can be reduced to the case 1).

Now, we introduce the following definitions.

Let  $I$  be an index set. For each  $i \in I$ , let  $X_i$  be a non-empty convex subset of a topological linear space  $E_i$  and denote  $X = \prod_{i \in I} X_i$ .

**Definition 6.** Let  $K_i$  be a subset of  $X$ . The correspondence  $A_i : X \rightarrow 2^{X_i}$  is said to have the WNQS-property on  $K_i$ , if there is a weakly naturally quasi-concave correspondence  $T_i : K_i \rightarrow 2^{X_i}$  such that  $x_i \notin T_i(x)$  and  $T_i(x) \subset A_i(x)$  for all  $x \in K_i$ .

**Definition 7.** Let  $K_i$  be a subset of  $X$ . The correspondence  $A_i : X \rightarrow 2^{X_i}$  is said to have the e-WNQS-property on  $K_i$  if for each convex neighborhood  $V$  of 0 in  $X_i$ , there is a weakly naturally quasi-concave correspondence  $T_i^V : K_i \rightarrow 2^{X_i}$  such that  $x_i \notin T_i^V(x)$  and  $T_i^V(x) \subset A_i(x) + V$  for all  $x \in K_i$ .

**Definition 8.** Let  $X$  be a nonempty convex subset of a topological vector space  $E$  and  $Y$  be a nonempty subset of a topological vector space  $F$ . The correspondence  $T : X \rightarrow 2^Y$  is said to be weakly \*-concave if for each  $n$  and for each finite set  $\{x_1, x_2, \dots, x_n\} \subset X$ , there exists  $y_i \in T(x_i)$ , ( $i = 1, 2, \dots, n$ ), such that for every  $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \Delta_{n-1}$ ,  $\sum_{i=1}^n \lambda_i y_i \in T(x)$ , for each  $x \in X$ .

To prove our theorems of equilibrium existence, we need the following:

**Theorem 3.** (Wu's fixed point theorem [19]) Let  $I$  be an index set. For each  $i \in I$ , let  $X_i$  be a nonempty convex subset of a Hausdorff locally convex topological vector space  $E_i$ ,  $D_i$  a non-empty compact metrizable subset of  $X_i$  and  $S_i, T_i : X := \prod_{i \in I} X_i \rightarrow 2^{D_i}$  two correspondences with the following conditions:

- (i) for each  $x \in X$ ,  $\text{clco} S_i(x) \subset T_i(x)$  and  $S_i(x) \neq \emptyset$ ,
- (ii)  $S_i$  is lower semicontinuous.

Then, there exists a point  $\bar{x} = \prod_{i \in I} \bar{x}_i \in D = \prod_{i \in I} D_i$  such that  $\bar{x}_i \in T_i(\bar{x})$  for each  $i \in I$ .

The extension of Kakutani's theorem on locally convex spaces is due to Ky Fan.

**Theorem 4.** (*Ky-Fan, [7]*) Let  $Y$  be a locally convex space,  $X \subset Y$  be a compact and convex subset and  $T : X \rightarrow 2^X$  be an upper semicontinuous correspondence with non-empty compact convex values. Then,  $T$  has a fixed point.

For the case when  $X$  is not compact, Himmelberg got the following result.

**Theorem 5.** (*Himmelberg, [8]*) Let  $X$  be a non-empty convex subset of a separated locally convex space  $Y$ . Let  $T : X \rightarrow 2^X$  be an upper semicontinuous correspondence such that  $T(x)$  is closed and convex for each  $x \in X$ , and  $T(X)$  is contained in a compact subset  $C$  of  $X$ . Then,  $T$  has a fixed point.

### 3. FIXED POINT THEOREMS

We formulate the following fixed point theorem for weakly naturally quasi-concave correspondences.

**Theorem 6.** (*selection theorem*) Let  $Y$  be a non-empty subset of a topological vector space  $E$  and  $K$  be a  $(n-1)$ -dimensional simplex in a topological vector space  $F$ . Let  $T : K \rightarrow 2^Y$  be a weakly naturally quasi-concave correspondence. Then,  $T$  has a continuous selection on  $K$ .

**Proof.** Let  $a_1, a_2, \dots, a_n$  be the vertices of  $K$ . Since  $T$  is weakly naturally quasi-concave, there exist  $b_i \in T(a_i)$ , ( $i = 1, 2, \dots, n$ ) and  $g = (g_1, g_2, \dots, g_n) : \Delta_{n-1} \rightarrow \Delta_{n-1}$  a function with  $g_i$  continuous,  $g_i(1) = 1$  and  $g_i(0) = 0$  for each  $i = 1, 2, \dots, n$ , such that for every  $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \Delta_{n-1}$ , there exists  $y \in T(\sum_{i=1}^n \lambda_i a_i)$  with  $y = \sum_{i=1}^n g_i(\lambda_i) y_i$ .

Since  $K$  is a  $(n-1)$ -dimensional simplex with the vertices  $a_1, \dots, a_n$ , there exists unique continuous functions  $\lambda_i : K \rightarrow \mathbb{R}$ ,  $i = 1, 2, \dots, n$  such that for each  $x \in K$ , we have  $(\lambda_1(x), \lambda_2(x), \dots, \lambda_n(x)) \in \Delta_{n-1}$  and  $x = \sum_{i=1}^n \lambda_i(x) a_i$ .

Let's define  $f : K \rightarrow 2^Y$  by

$f(a_i) = b_i$  ( $i = 1, \dots, n$ ) and

$f(\sum_{i=1}^n \lambda_i a_i) = \sum_{i=1}^n g_i(\lambda_i) b_i \in T(x)$ .

We show that  $f$  is continuous.



Let  $(x_m)_{m \in \mathbb{N}}$  be a sequence which converges to  $x_0 \in K$ , where  $x_m = \sum_{i=1}^n \lambda_i(x_m) a_i$  and  $x_0 = \sum_{i=1}^n \lambda_i(x_0) a_i$ . By the continuity of  $\lambda_i$ , it follows that for each  $i = 1, 2, \dots, n$ ,  $\lambda_i(x_m) \rightarrow \lambda_i(x_0)$  as  $m \rightarrow \infty$ . Since  $g_1, \dots, g_n$  are continuous, we have  $g_i(\lambda_i(x_m)) \rightarrow g_i(\lambda_i(x_0))$  as  $m \rightarrow \infty$ . Hence  $f(x_m) \rightarrow f(x_0)$  as  $m \rightarrow \infty$ , i.e.  $f$  is continuous. ■

**Theorem 7.** *Let  $Y$  be a non-empty subset of a topological vector space  $E$  and  $K$  be a  $(n-1)$ -dimensional simplex in  $E$ . Let  $T : K \rightarrow 2^Y$  be an weakly naturally quasi-concave correspondence and  $s : Y \rightarrow K$  be a continuous function. Then, there exists  $x^* \in K$  such that  $x^* \in s \circ T(x^*)$ .*

**Proof.** By Theorem 6,  $T$  has a continuous selection theorem on  $K$ . Since  $s : Y \rightarrow K$  is continuous, we obtain that  $s \circ f : K \rightarrow K$  is continuous. By Brouwer's fixed point theorem, there exists a point  $x^* \in K$  such that  $x^* = s \circ f(x^*)$  and then,  $x^* \in s \circ T(x^*)$ . ■

**Theorem 8.** *(selection theorem). Let  $Y$  be a non-empty subset of a topological vector space  $E$  and  $K$  be a  $(n-1)$ -dimensional simplex in a topological vector space  $F$ . Let  $T : K \rightarrow 2^Y$  be a weakly \*-concave correspondence. Then,  $T$  has a continuous selection on  $K$ .*

**Proof.** Let  $a_1, a_2, \dots, a_n$  be the vertices of  $K$ . Since  $T$  is weakly \*-concave, there exist  $b_i \in T(a_i)$ , ( $i = 1, 2, \dots, n$ ) such that for every  $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \Delta_{n-1}$ ,  $\sum_{i=1}^n \lambda_i b_i \subset T(x)$ , for each  $x \in K$ .

Since  $K$  is a  $(n-1)$ -dimensional simplex with the vertices  $a_1, \dots, a_n$ , there exists unique continuous functions  $\lambda_i : K \rightarrow \mathbb{R}$ ,  $i = 1, 2, \dots, n$  such that for each  $x \in K$ , we have  $(\lambda_1(x), \lambda_2(x), \dots, \lambda_n(x)) \in \Delta_{n-1}$  and  $x = \sum_{i=1}^n \lambda_i(x) a_i$ .

Let's define  $f : K \rightarrow 2^Y$  by

$f(a_i) = b_i$  ( $i = 1, \dots, n$ ) and

$f(\sum_{i=1}^n \lambda_i a_i) = \sum_{i=1}^n \lambda_i b_i \in T(x)$ .

We show that  $f$  is continuous.

Let  $(x_m)_{m \in \mathbb{N}}$  be a sequence which converges to  $x_0 \in K$  where  $x_m = \sum_{i=1}^n \lambda_i(x_m) a_i$  and  $x_0 = \sum_{i=1}^n \lambda_i(x_0) a_i$ . By the continuity of  $\lambda_i$ , it follows that

for each  $i = 1, 2, \dots, n$ ,  $\lambda_i(x_m) \rightarrow \lambda_i(x_0)$  as  $m \rightarrow \infty$ . Hence we must have  $f(x_m) \rightarrow f(x_0)$  as  $m \rightarrow \infty$ , i.e.  $f$  is continuous. ■

**Theorem 9.** *Let  $Y$  be a non-empty subset of a topological vector space  $E$  and  $K$  be a  $(n - 1)$ - dimensional simplex in  $E$ . Let  $T : K \rightarrow 2^Y$  be a weakly  $*$ -concave correspondence and  $s : Y \rightarrow K$  be a continuous function. Then, there exists  $x^* \in K$  such that  $x^* \in s \circ T(x^*)$ .*

**Proof.** By Theorem 8,  $T$  has a continuous selection theorem on  $K$ . Since  $s : Y \rightarrow K$  is continuous, we obtain that  $s \circ f : K \rightarrow K$  is continuous. By Brouwer's fixed point theorem, there exists a point  $x^* \in K$  such that  $x^* = s \circ f(x^*)$  and then,  $x^* \in s \circ T(x^*)$ . ■

#### 4. EQUILIBRIUM THEOREMS

First, we present the model of an abstract economy and the definition of an equilibrium.

Let  $I$  be a non-empty set (the set of agents). For each  $i \in I$ , let  $X_i$  be a non-empty topological vector space representing the set of actions and define  $X := \prod_{i \in I} X_i$ ; let  $A_i, B_i : X \rightarrow 2^{X_i}$  be the constraint correspondences and  $P_i$  the preference correspondence.

**Definition 9.** *The family  $\Gamma = (X_i, A_i, P_i, B_i)_{i \in I}$  is said to be an abstract economy.*

**Definition 10.** *An equilibrium for  $\Gamma$  is defined as a point  $\bar{x} \in X$  such that for each  $i \in I$ ,  $\bar{x}_i \in \overline{B_i}(\bar{x})$  and  $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$ .*

**Remark 3.** *When for each  $i \in I$ ,  $A_i(x) = B_i(x)$  for all  $x \in X$ , this abstract economy model coincides with the classical one introduced by Borglin and Keiding in [2]. If in addition,  $\overline{B_i}(\bar{x}) = \text{cl}_{X_i} B_i(\bar{x})$  for each  $x \in X$ , which is the case if  $B_i$  has a closed graph in  $X \times X_i$ , the definition of an equilibrium coincides with the one used by Yannelis and Prabhakar [21].*

To prove the following theorems we use the selection theorem mentioned in Section 3. We show the existence of equilibrium for an abstract economy without assuming the continuity of the constraint and the preference correspondences  $A_i$  and  $P_i$ .

First, we prove a new equilibrium existence theorem for a noncompact abstract economy with constraint and preference correspondences  $A_i$  and  $P_i$ , which have the property that their intersection  $A_i \cap P_i$  contains a WNQ selector on the domain  $W_i$  of  $A_i \cap P_i$  and  $W_i$  must be a simplex. To find the equilibrium point, we use Wu's fixed point theorem [19].

Since the constraint correspondence  $B_i$  is lower semicontinuous for each  $i \in I$ , the next theorem can be compared with Theorem 5 of Wu [19]. The proofs of these results are based on similar methods.

**Theorem 10.** *Let  $\Gamma = (X_i, A_i, P_i, B_i)_{i \in I}$  be an abstract economy, where  $I$  is a (possibly uncountable) set of agents such that for each  $i \in I$  :*

- (1)  $X_i$  is a non-empty convex set in a locally convex space  $E_i$  and there exists a compact subset  $D_i$  of  $X_i$  containing all the values of the correspondences  $A_i, P_i$  and  $B_i$  such that  $D = \prod_{i \in I} D_i$  is metrizable;
- (2)  $\text{cl}B_i$  is lower semicontinuous, has non-empty convex values and for each  $x \in X$ ,  $A_i(x) \subset B_i(x)$ ;
- (3)  $W_i = \{x \in X \mid (A_i \cap P_i)(x) \neq \emptyset\}$  is a  $(n_i - 1)$ -dimensional simplex in  $X$  such that  $W_i \subset \text{co}D$ ;
- (4) there exists a weakly naturally quasi-concave correspondence  $S_i : W_i \rightarrow 2^{D_i}$  such that  $S_i(x) \subset (A_i \cap P_i)(x)$  for each  $x \in W_i$ ;
- (5) for each  $x \in W_i$ ,  $x_i \notin (A_i \cap P_i)(x)$ .

*Then, there exists an equilibrium point  $\bar{x} \in D$  for  $\Gamma$ , i.e., for each  $i \in I$ ,  $\bar{x}_i \in \text{cl}B_i(\bar{x})$  and  $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$ .*

**Proof.** Let be  $i \in I$ . From the assumption (4) and the selection theorem (Theorem 6), it follows that there exists a continuous function  $f_i : W_i \rightarrow D_i$  such that for each  $x \in W_i$ ,  $f_i(x) \in S_i(x) \subset A_i(x) \cap P_i(x) \subset B_i(x)$ .

Define the correspondence  $T_i : X \rightarrow 2^{D_i}$ , by  $T_i(x) := \begin{cases} \{f_i(x)\}, & \text{if } x \in W_i, \\ \text{cl}B_i(x), & \text{if } x \notin W_i. \end{cases}$

$T_i$  is lower semicontinuous on  $X$ .

Let  $V$  be a closed subset of  $X_i$ , then

$$U := \{x \in X \mid T_i(x) \subset V\} = \{x \in W_i \mid T_i(x) \subset V\} \cup \{x \in X \setminus W_i \mid T_i(x) \subset V\}$$

$$\begin{aligned}
&= \{x \in W_i \mid f_i(x) \in V\} \cup \{x \in X \mid \text{cl}B_i(x) \subset V\} \\
&= (f_i^{-1}(V) \cap W_i) \cup \{x \in X \mid \text{cl}B_i(x) \subset V\}.
\end{aligned}$$

$U$  is a closed set, because  $W_i$  is closed,  $f_i$  is a continuous function on  $\text{int}_X K_i$  and the set  $\{x \in X \mid \text{cl}B_i(x) \subset V\}$  is closed since  $\text{cl}B_i$  is l.s.c. Let  $D = \prod_{i \in I} D_i$ . Then by Tychonoff's Theorem,  $D$  is compact in the convex set  $X$ .

By Theorem 3 (Wu's fixed-point theorem), applied for the correspondences  $S_i = T_i$  and  $T_i : X \rightarrow 2^{D_i}$ , there exists  $\bar{x} \in D$  such that for each  $i \in I$ ,  $\bar{x}_i \in T_i(\bar{x})$ . If  $\bar{x} \in W_i$  for some  $i \in I$ , then  $\bar{x}_i = f_i(\bar{x})$ , which is a contradiction.

Therefore,  $\bar{x} \notin W_i$ , and hence,  $(A_i \cap P_i)(\bar{x}) = \emptyset$ . Also, for each  $i \in I$ , we have  $\bar{x}_i \in T_i(\bar{x})$ , and then,  $\bar{x}_i \in \text{cl}B_i(\bar{x})$ . ■

**Remark 4.** *In this theorem, the correspondences  $A_i \cap P_i$ ,  $i \in I$ , may not verify continuity assumptions and may not have convex or compact values.*

**Remark 5.** *In assumption (3),  $W_i$  must be a proper subset of  $X$ . In fact, if  $W_i = X_i$ , then, by applying Himmelberg's fixed point theorem ([8]) to  $\prod_{i \in I} f_i(x)$ , where  $f_i$  is a continuous selection of  $S_i \subset A_i \cap P_i$ , we can get a fixed point  $\bar{x} \in \prod_{i \in I} (A_i \cap P_i)(\bar{x})$ , which contradicts assumption (5).*

Since a correspondence  $T : X \rightarrow 2^Y$  having the property that  $\cap\{T(x) : x \in X\}$  is nonempty and convex, is a WNQ correspondence, we obtain the following corollary.

**Corollary 11.** *Let  $\Gamma = (X_i, A_i, P_i, B_i)_{i \in I}$  be an abstract economy, where  $I$  is a (possibly uncountable) set of agents such that for each  $i \in I$  :*

- (1)  $X_i$  is a non-empty convex set in a locally convex space  $E_i$  and there exists a compact subset  $D_i$  of  $X_i$  containing all the values of the correspondences  $A_i, P_i$  and  $B_i$  such that  $D = \prod_{i \in I} D_i$  is metrizable;
- (2)  $\text{cl}B_i$  is lower semicontinuous, has non-empty convex values and for each  $x \in X$ ,  $A_i(x) \subset B_i(x)$ ;
- (3)  $W_i = \{x \in X \mid (A_i \cap P_i)(x) \neq \emptyset\}$  is a  $(n_i - 1)$ -dimensional simplex in  $X$  such that  $W_i \subset \text{co}D$ ;

(4) *there exists a correspondence  $S_i : W_i \rightarrow 2^{D_i}$  such that  $S_i$  has the property that  $\cap\{T(x) : x \in X\}$  is nonempty and convex, and  $S_i(x) \subset (A_i \cap P_i)(x)$  for each  $x \in W_i$ ;*

(5) *for each  $x \in W_i$ ,  $x_i \notin (A_i \cap P_i)(x)$ .*

*Then there exists an equilibrium point  $\bar{x} \in D$  for  $\Gamma$ , i.e., for each  $i \in I$ ,  $\bar{x}_i \in \text{cl}B_i(\bar{x})$  and  $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$ .*

A correspondence  $T : X \rightarrow 2^Y$  with convex graph is a WNQ correspondence, and then we have:

**Corollary 12.** *Let  $\Gamma = (X_i, A_i, P_i, B_i)_{i \in I}$  be an abstract economy, where  $I$  is a (possibly uncountable) set of agents such that for each  $i \in I$ :*

(1)  *$X_i$  is a non-empty compact convex set in a locally convex space  $E_i$ ;*

(2)  *$\text{cl}B_i$  is lower semicontinuous, has non-empty convex values and for each  $x \in X$ ,  $A_i(x) \subset B_i(x)$ ;*

(3)  *$W_i = \{x \in X / (A_i \cap P_i)(x) \neq \emptyset\}$  is a  $(n_i - 1)$ -dimensional simplex in  $X$ ;*

(4) *there exists a correspondence  $S_i : W_i \rightarrow 2^{X_i}$  with convex graph such that  $S_i(x)$*

*$\subset (A_i \cap P_i)(x)$  for each  $x \in W_i$ ;*

(5) *for each  $x \in W_i$ ,  $x_i \notin (A_i \cap P_i)(x)$ .*

*Then there exists an equilibrium point  $\bar{x} \in X$  for  $\Gamma$ , i.e., for each  $i \in I$ ,  $\bar{x}_i \in \text{cl}B_i(\bar{x})$  and  $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$ .*

For Theorem 13, we use an approximation method, in the meaning that we obtain, for each  $i \in I$ , a continuous selection  $f_i^{V_i}$  of  $(A_i + V_i) \cap P_i$ , where  $V_i$  is a convex neighborhood of 0 in  $X_i$ . For every  $V = \prod_{i \in I} V_i$ , we obtain an equilibrium point for the associated approximate abstract economy  $\Gamma_V = (X_i, A_i, P_i, B_{V_i})_{i \in I}$ , i.e., a point  $\bar{x} \in X$  such that  $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$  and  $\bar{x}_i \in B_{V_i}(\bar{x})$ , where the correspondence  $B_{V_i} : X \rightarrow 2^{X_i}$  is defined by  $B_{V_i}(x) = \text{cl}(B_i(x) + V_i) \cap X_i$  for each  $x \in X$  and for each  $i \in I$ . Finally, we use Lemma 1 to get an equilibrium point for  $\Gamma$  in  $X$ . The compactness assumption for  $X_i$  is essential in the proof.

Examples of results which use an approximation method are Theorem 3.1 pag 37 or Theorem 1.2, pag. 41 in [23]. This method is usually used in relation with abstract economies which have lower semicontinuous constraint correspondences.

**Theorem 13.** Let  $\Gamma = (X_i, A_i, P_i, B_i)_{i \in I}$  be an abstract economy, where  $I$  is a (possibly uncountable) set of agents such that for each  $i \in I$  :

- (1)  $X_i$  is a non-empty compact convex set in a locally convex space  $E_i$ ;
- (2)  $\text{cl}B_i$  is upper semicontinuous, has non-empty convex values and for each  $x \in X$ ,  $A_i(x) \subset B_i(x)$ ;
- (3) the set  $W_i := \{x \in X / (A_i \cap P_i)(x) \neq \emptyset\}$  is non-empty, open and  $K_i = \text{cl}W_i$  is a  $(n_i - 1)$ -dimensional simplex in  $X$ ;
- (4) For each convex neighbourhood  $V$  of 0 in  $X_i$ ,  $(A_i + V) \cap P_i : K_i \rightarrow 2^{X_i}$  is a weakly naturally quasi-concave correspondence;
- (5) for each  $x \in K_i$ ,  $x_i \notin P_i(x)$ .

Then there exists an equilibrium point  $\bar{x} \in X$  for  $\Gamma$ , i.e., for each  $i \in I$ ,  $\bar{x}_i \in \bar{B}_i(\bar{x})$  and  $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$ .

**Proof.** For each  $i \in I$ , let  $\beta_i$  denote the family of all open convex neighborhoods of zero in  $E_i$ . Let  $V = (V_i)_{i \in I} \in \prod_{i \in I} \beta_i$ . Since  $(A_i + V_i) \cap P_i$  is a weakly concave like correspondence on  $K_i$ , then, from the selection theorem (Theorem 6), there exists a continuous function  $f_i^{V_i} : K_i \rightarrow X_i$  such that for each  $x \in K_i$ ,

$$f_i^{V_i}(x) \in (A_i(x) + V_i) \cap P_i(x) \subset (A_i(x) + V_i) \cap X_i.$$

It follows that  $f_i^{V_i}(x) \in \text{cl}(B_i(x) + V_i)$  for  $x \in K_i$ . Since  $X_i$  is compact, we have that  $\text{cl}B_i(x)$  is compact for every  $x \in X$  and  $\text{cl}(B_i(x) + V_i) = \text{cl}(B_i(x)) + \text{cl}V_i$  for every  $V_i \subset E_i$ .

Define the correspondence  $T_i^{V_i} : X \rightarrow 2^{X_i}$ , by

$$T_i^{V_i}(x) := \begin{cases} \{f_i^{V_i}(x)\}, & \text{if } x \in \text{int}_X K = W_i, \\ \text{cl}(B_i(x) + V_i) \cap X_i, & \text{if } x \in X \setminus \text{int}_X K_i; \end{cases}$$

The correspondence  $B_{V_i} : X \rightarrow 2^{X_i}$ , defined by  $B_{V_i}(x) := \text{cl}(B_i(x) + V_i) \cap X_i$  is u.s.c. by Lemma 2. Then following the same line as in Theorem 10, we can prove that  $T_i^{V_i}$  is upper semicontinuous on  $X$  and has closed convex values.

Define  $T^V : X \rightarrow 2^X$  by  $T^V(x) := \prod_{i \in I} T_i^{V_i}(x)$  for each  $x \in X$ .

$T^V$  is an upper semicontinuous correspondence and it also has non-empty convex closed values.

Since  $X$  is a compact convex set, by Fan's fixed-point theorem [7], there exists  $\bar{x}_V \in X$  such that  $\bar{x}_V \in T^V(\bar{x}_V)$ , i.e., for each  $i \in I$ ,  $(\bar{x}_V)_i \in T_i^{V_i}(\bar{x}_V)$ .

We state that  $\bar{x}_V \in X \setminus \bigcup_{i \in I} \text{int}_X K_i$ .

If  $\bar{x}_V \in \text{int}_X K_i$ ,  $(\bar{x}_V)_i \in T_i^{V_i}(\bar{x}_V) = f_i(\bar{x}_V) \in ((A_i(\bar{x}_V) + V_i) \cap P_i)(\bar{x}_V) \subset P_i(\bar{x}_V)$ , which contradicts assumption (5).

Hence  $(\bar{x}_V)_i \in \text{cl}(B_i(\bar{x}_V) + V_i) \cap X_i$  and  $(A_i \cap P_i)(\bar{x}_V) = \emptyset$ , i.e.  $\bar{x}_V \in Q_V$  where

$$Q_V = \bigcap_{i \in I} \{x \in X : x_i \in \text{cl}(B_i(x) + V_i) \cap X_i \text{ and } (A_i \cap P_i)(x) = \emptyset\}.$$

Since  $W_i$  is open,  $Q_V$  is the intersection of non-empty closed sets, then it is non-empty, closed in  $X$ .

We prove that the family  $\{Q_V : V \in \prod_{i \in I} \beta_i\}$  has the finite intersection property.

Let  $\{V^{(1)}, V^{(2)}, \dots, V^{(n)}\}$  be any finite set of  $\prod_{i \in I} \beta_i$  and let  $V^{(k)} = (V_i^{(k)})_{i \in I}$ ,  $k = 1, \dots, n$ . For each  $i \in I$ , let  $V_i = \bigcap_{k=1}^n V_i^{(k)}$ , then  $V_i \in \beta_i$ ; thus  $V = (V_i)_{i \in I} \in \prod_{i \in I} \beta_i$ . Clearly  $Q_V \subset \bigcap_{k=1}^n Q_{V^{(k)}}$  so that  $\bigcap_{k=1}^n Q_{V^{(k)}} \neq \emptyset$ .

Since  $X$  is compact and the family  $\{Q_V : V \in \prod_{i \in I} \beta_i\}$  has the finite intersection property, we have that  $\bigcap_{i \in I} Q_V \neq \emptyset$ . Take any  $\bar{x} \in \bigcap_{i \in I} Q_V$ , then for each  $i \in I$  and each  $V_i \in \beta_i$ ,  $\bar{x}_i \in \text{cl}(B_i(\bar{x}) + V_i) \cap X_i$  and  $(A_i \cap P_i)(\bar{x}) = \emptyset$ ; but then  $\bar{x}_i \in \text{cl}(B_i(\bar{x}))$  by Lemma 1 and  $(A_i \cap P_i)(\bar{x}) = \emptyset$  for each  $i \in I$  so that  $\bar{x}$  is an equilibrium point of  $\Gamma$  in  $X$ . ■

The last two theorems can be compared with Zheng's theorems 3.1 and 3.2 in [24] and Zhou's theorems 5 and 6 in [25] where the constraint correspondences have continuous selections on a closed subset  $C_i \subset X$  which contains the set  $\{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$ .

To find the equilibrium point in Theorem 14, we use Wu's fixed point theorem for correspondences  $\text{cl}B_i$  which are lower semicontinuous and we need a non-empty compact metrizable set  $D_i$  in  $X_i$  for each  $i \in I$ . The spaces  $X_i$  are not compact.

**Theorem 14.** *Let  $\Gamma = (X_i, A_i, P_i, B_i)_{i \in I}$  be an abstract economy, where  $I$  is a (possibly uncountable) set of agents such that for each  $i \in I$ :*

- (1)  $X_i$  is a non-empty convex set in a Hausdorff locally convex space  $E_i$  and there exists a nonempty compact metrizable subset  $D_i$  of  $X_i$  containing all values of the correspondences  $A_i, P_i$  and  $B_i$ ;
- (2)  $\text{cl}B_i$  is lower semicontinuous with non-empty convex values;

(3) there exists a  $(n_i - 1)$ -dimensional simplex  $K_i$  in  $X$  and

$$W_i := \{x \in X \mid (A_i \cap P_i)(x) \neq \emptyset\} \subset \text{int}_X(K_i);$$

(4)  $\text{cl}B_i$  has the (WNQS)-property on  $K_i$ ;

Then there exists an equilibrium point  $\bar{x} \in D$  for  $\Gamma$ , i.e., for each  $i \in I$ ,  $\bar{x}_i \in \text{cl}B_i(\bar{x})$  and  $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$ .

**Proof.** *Proof.* Since  $\text{cl}B_i$  has the WNQS property on  $K_i$ , it follows that there exists a weakly concave like correspondence  $F_i : X \rightarrow 2^{D_i}$  such that  $F_i(x) \subset \text{cl}B_i(x)$  and  $x_i \notin F_i(x)$  for each  $x \in K_i$ .

$K_i$  is a  $(n_i - 1)$ -dimensional simplex, then, from the selection theorem, there exists a continuous function  $f_i : K_i \rightarrow D_i$  such that  $f_i(x) \in F_i(x)$  for each  $x \in K_i$ . Because  $x_i \notin F_i(x)$  for each  $x \in K_i$ , we have that  $x_i \neq f_i(x)$  for each  $x \in K_i$ .

Define the correspondence  $T_i : X \rightarrow 2^{D_i}$ , by  $T_i(x) := \begin{cases} \{f_i(x)\}, & \text{if } x \in K_i, \\ \text{cl}B_i(x), & \text{if } x \notin K_i. \end{cases}$

$T_i$  is lower semicontinuous on  $X$  and has closed convex values.

Let  $U$  be a closed subset of  $X_i$ , then

$$\begin{aligned} U' &:= \{x \in X \mid T_i(x) \subset U\} = \{x \in K_i \mid T_i(x) \subset U\} \cup \{x \in X \setminus K_i \mid \\ &T_i(x) \subset U\} \\ &= \{x \in K_i \mid f_i(x) \in U\} \cup \{x \in X \mid \text{cl}B_i(x) \subset U\} \\ &= ((f_i)^{-1}(U) \cap K_i) \cup \{x \in X \mid \text{cl}B_i(x) \subset U\}. \end{aligned}$$

$U'$  is a closed set, because  $K_i$  is closed,  $f_i$  is a continuous function on  $K_i$  and the set  $\{x \in X \mid \text{cl}B_i(x) \subset U\}$  is closed since  $\text{cl}B_i(x)$  is l.s.c. Then  $T_i$  is lower semicontinuous on  $X$  and has non-empty closed convex values.

By Theorem 3 (Wu's fixed-point theorem) applied for the correspondences  $S_i = T_i$  and  $T_i : X \rightarrow 2^{D_i}$ , there exists  $\bar{x} \in D$  such that for each  $i \in I$ ,  $\bar{x}_i \in T_i(\bar{x})$ . If  $\bar{x} \in W_i$  for some  $i \in I$ , then  $\bar{x}_i = f_i(\bar{x})$ , which is a contradiction.

Therefore,  $\bar{x} \notin W_i$ , and hence  $(A_i \cap P_i)(\bar{x}) = \emptyset$ . Also, for each  $i \in I$ , we have  $\bar{x}_i \in T_i(\bar{x})$ , and then  $\bar{x}_i \in \text{cl}B_i(\bar{x})$ . ■

In Theorem 15 the sets  $X_i$  are non-empty compact convex in locally convex spaces  $E_i$ . As in Theorem 13, we first obtain equilibria for  $\Gamma_V$ , and then, the proof coincides with the proof of Theorem 13.

**Theorem 15.** Let  $\Gamma = (X_i, A_i, P_i, B_i)_{i \in I}$  be an abstract economy, where  $I$  is a (possibly uncountable) set of agents such that for each  $i \in I$  :

- (1)  $X_i$  is a non-empty compact convex set in a locally convex space  $E_i$ ;
- (2)  $\text{cl}B_i$  is upper semicontinuous with non-empty convex values;



(3) the set  $W_i := \{x \in X \mid (A_i \cap P_i)(x) \neq \emptyset\}$  is open and there exists

a

$(n_i - 1)$ -dimensional simplex  $K_i$  in  $X$  such that  $W_i \subset \text{int}_X(K_i)$ .

(3)  $\text{cl}B_i$  has the (e-WNQS)-property on  $K_i$ .

Then there exists an equilibrium point  $\bar{x} \in X$  for  $\Gamma$ , i.e., for each  $i \in I$ ,  $\bar{x}_i \in \bar{B}_i(\bar{x})$  and  $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$ .

**Proof.** For each  $i \in I$ , let  $\beta_i$  denote the family of all open convex neighborhoods of zero in  $E_i$ . Let  $V = (V_i)_{i \in I} \in \prod_{i \in I} \beta_i$ . Since  $\text{cl}B_i$  has the e-WNQS property on  $K_i$ , it follows that there exists a weakly concave like correspondence  $F_i^{V_i} : X \rightarrow 2^{X_i}$  such that  $F_i^{V_i}(x) \subset \text{cl}B_i(x) + V_i$  and  $x_i \notin F_i^{V_i}(x)$  for each  $x \in K_i$ .

$K_i$  is a  $(n_i - 1)$ -dimensional simplex, then, from the selection theorem, there exists a continuous function  $f_i^{V_i} : K_i \rightarrow X_i$  such that  $f_i^{V_i}(x) \in F_i^{V_i}(x)$  for each  $x \in K_i$ . Because  $x_i \notin F_i^{V_i}(x)$  for each  $x \in K_i$ , we have that  $x_i \neq f_i^{V_i}(x)$  for each  $x \in K_i$ .

Define the correspondence  $T_i^{V_i} : X \rightarrow 2^{X_i}$ , by

$$T_i^{V_i}(x) := \begin{cases} \{f_i^{V_i}(x)\}, & \text{if } x \in \text{int}_X K_i, \\ \text{cl}(B_i(x) + V_i) \cap X_i, & \text{if } x \in X \setminus \text{int}_X K_i; \end{cases}$$

$B_{V_i} : X \rightarrow 2^{X_i}$ ,  $B_{V_i}(x) = \text{cl}(B_i(x) + V_i) \cap X_i = (\text{cl}B_i(x) + \text{cl}V_i) \cap X_i$  is upper semicontinuous by Lemma 2.

Let  $U$  be an open subset of  $X_i$ , then

$$\begin{aligned} U' &:= \{x \in X \mid T_i^{V_i}(x) \subset U\} \\ &= \{x \in \text{int}_X K_i \mid T_i^{V_i}(x) \subset U\} \cup \{x \in X \setminus \text{int}_X K_i \mid T_i^{V_i}(x) \subset U\} \\ &= \{x \in \text{int}_X K_i \mid f_i^{V_i}(x, y) \in U\} \cup \{x \in X \mid (\text{cl}B_i(x) + \bar{V}_i) \cap X_i \subset U\} \\ &= ((f_i^{V_i})^{-1}(U) \cap \text{int}_K K_i) \cup \{x \in X \mid (\text{cl}B_i(x) + \bar{V}_i) \cap X_i \subset U\}. \end{aligned}$$

$U'$  is an open set, because  $\text{int}_X K_i$  is open,  $f_i^{V_i}$  is a continuous function on  $K_i$  and the set  $\{x \in X \mid (\text{cl}B_i(x) + \text{cl}V_i) \cap X_i \subset U\}$  is open since  $(\text{cl}B_i(x) + \text{cl}V_i) \cap X_i$  is u.s.c. Then  $T_i^{V_i}$  is upper semicontinuous on  $X$  and has closed convex values.

Define  $T^V : X \rightarrow 2^X$  by  $T^V(x) := \prod_{i \in I} T_i^{V_i}(x)$  for each  $x \in X$ .

$T^V$  is an upper semicontinuous correspondence and it has also non-empty convex closed values.

Since  $X$  is a compact convex set, by Fan's fixed-point theorem [7], there exists  $\bar{x}_V \in X$  such that  $\bar{x}_V \in T^V(\bar{x}_V)$ , i.e., for each  $i \in I$ ,  $(\bar{x}_V)_i \in T_i^{V_i}(\bar{x}_V)$ . If  $\bar{x}_V \in \text{int}_X K_i$ ,  $(\bar{x}_V)_i = f_i^{V_i}(\bar{x}_V)$ , which is a contradiction.

Hence  $(\bar{x}_V)_i \in \text{cl}(B_i(\bar{x}_V) + V_i) \cap X_i$  and  $(A_i \cap P_i)(\bar{x}_V) = \emptyset$ , i.e.  $\bar{x}_V \in Q_V$  where

$$Q_V = \cap_{i \in I} \{x \in X : x_i \in \text{cl}(B_i(x) + V_i) \cap X_i \text{ and } (A_i \cap P_i)(x) = \emptyset\}.$$

Since  $W_i$  is open,  $Q_V$  is the intersection of non-empty closed sets, then it is non-empty, closed in  $X$ .

We prove that the family  $\{Q_V : V \in \prod_{i \in I} \mathcal{B}_i\}$  has the finite intersection property.

Let  $\{V^{(1)}, V^{(2)}, \dots, V^{(n)}\}$  be any finite set of  $\prod_{i \in I} \mathcal{B}_i$  and let  $V^{(k)} = (V_i^{(k)})_{i \in I}$ ,  $k = 1, \dots, n$ . For each  $i \in I$ , let  $V_i = \bigcap_{k=1}^n V_i^{(k)}$ , then  $V_i \in \mathcal{B}_i$ ; thus  $V = (V_i)_{i \in I} \in \prod_{i \in I} \mathcal{B}_i$ . Clearly  $Q_V \subset \bigcap_{k=1}^n Q_{V^{(k)}}$  so that  $\bigcap_{k=1}^n Q_{V^{(k)}} \neq \emptyset$ . Since  $X$  is compact and the family  $\{Q_V : V \in \prod_{i \in I} \mathcal{B}_i\}$  has the finite intersection property, we have that  $\cap \{Q_V : V \in \prod_{i \in I} \mathcal{B}_i\} \neq \emptyset$ . Take any  $\bar{x} \in \cap \{Q_V : V \in \prod_{i \in I} \mathcal{B}_i\}$ , then for each  $i \in I$  and each  $V_i \in \mathcal{B}_i$ ,  $\bar{x}_i \in \text{cl}(B_i(\bar{x}) + V_i) \cap X_i$  and  $(A_i \cap P_i)(\bar{x}) = \emptyset$ ; but then  $\bar{x}_i \in \text{cl}(B_i(\bar{x}))$  from Lemma 1 and  $(A_i \cap P_i)(\bar{x}) = \emptyset$  for each  $i \in I$  so that  $\bar{x}$  is an equilibrium point of  $\Gamma$  in  $X$ . ■

This work was supported by the strategic grant POSDRU/89/1.5/S/58852, Project "Postdoctoral programme for training scientific researchers" cofinanced by the European Social Found within the Sectorial Operational Program Human Resources Development 2007-2013.

## References

- [1] H. Ben-El-Mechaiekh, *Fixed points for compact set-valued maps*, Questions Answers Gen. Topology. 10(1992), 153-156.
- [2] A. Borglin and H. Keiding, *Existence of equilibrium action and of equilibrium: A note on the 'new' existence theorem*, J. Math. Econom. 3(1976), 313-316.
- [3] F. E. Browder, *A new generation of the Schauder fixed point theorems*, Math. Ann. 174(1967), 285-290.
- [4] F. E. Browder, *The fixed point theory of multi-valued mappings in topological vector spaces*, Math. Ann. 177(1968), 283-301.

- [5] X. Ding and He Yiran, *Best Approximation Theorem for Set-valued Mappings without Convex Values and Continuity*, Appl Math. and Mech. English Edition, 19, 9 (1998), 831-836.
- [6] X. P. Ding, W. K. Kim and K. K. Tan, *A selection theorem and its applications*, Bull. Austral. Math. Soc. 46(1992), 205-212.
- [7] K. Fan, *Fixed-point and minimax theorems in locally convex topological linear spaces*, Proc. Nat. Acad. Sci. U.S.A. 38(1952), 121-126.
- [8] C. J. Himmelberg, *Fixed points of compact multifunctions*, J.Math. Anal. Appl. 38(1972), 205-207.
- [9] C. D. Horvath, *Extension and selection theorems in topological vector spaces with a generalized convexity structure*, Ann. Fac. Sci., Toulouse 2(1993), 253-269.
- [10] T. Husain and E. Taradfar, *A selection theorem and a fixed point theorem and an equilibrium point of an abstract economy*, Internat. J. Math. Math. Sci. 18(1995), 179-184.
- [11] E. Michael, *Continuous selection*, Annals of Mathematics 63, 2(1956), 361-382.
- [12] S. Park, *Continuous selection theorems in generalized convex spaces*, Numer. Funct. Anal. Optim. 25(1999), 567-583.
- [13] S. Park, *The Knaster-Kuratowski-Mazurkiewicz Theorem and almost fixed points*. Top. Methods in Nonlinear Anal, 16(2000), 195-200.
- [14] W. Shafer and H. Sonnenschein, *Equilibrium in abstract economies without ordered preferences*, Journal of Mathematical Economics 2(1975), 345-348.
- [15] T. Tanaka, *Generalized Quasiconvexities, Cone Saddle Points, and Minimax Theorem for Vector-Valued Functions*, J. Optim. Theory Appl. 81 2, (1994), 355-377.
- [16] G. Tian, *Fixed points theorems for mappings with noncompact and non-convex domains*, J. Math. Anal. Appl. 158(1991) 161-167.

- [17] K. Włodarczyk, D. Klim, *Fixed point and coincidence theorems for set-valued maps with nonconvex or noncompact domains in topological vector spaces*, Abstr. Appl. Anal. 1(2003) 1-18.
- [18] K. Włodarczyk, D. Klim, *Equilibria and fixed points of sets-valued maps with nonconvex and noncompact domains and ranges*, Nonlinear Analysis 65(2006), 918-932.
- [19] X. Wu, *A new fixed point theorem and its applications*, Proc. Amer. Math. Soc. 125(1997), 1779-1783.
- [20] X. Wu and S. Shen, *A further generalization of Yannelis-Prabhakar's continuous selection theorem and its applications*, J. Math. Anal. Appl. 197(1996), 61-74.
- [21] N. C. Yannelis and N. D. Prabhakar, *Existence of maximal elements and equilibrium in linear topological spaces*, J. Math. Econom. 12(1983), 233-245.
- [22] Z. T. Yu and L. J. Lin, *Continuous selection and fixed point theorems*, Nonlinear Anal. 52(2003), 445-455.
- [23] X. Z. Yuan, *The Study of Minimax Inequalities and Applications to Economies and Variational Inequalities*, Memoirs of the American Society, 625(1988).
- [24] X. Zheng, *Approximate selection theorems and their applications*, J. Math. An. Appl. 212(1997), 88-97.
- [25] J. Zhou, *On the Existence of equilibrium for abstract economies*. J. Math. Anal. Appl., 193(1992), 839-857.
- [26] Q. Zhang, C. Cheng and X. Li, *Generalized minimax theorems for two set-valued mappings*, Journal of Industrial and Management Optimization, 9 1 (2013), 1-12.